

# Noncommutative Probability, Matrix Models, and Quantum Orbifold Spacetimes

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## Abstract

Inspired by the intimate relationship between Voiculescu's noncommutative probability (of type A) and large- $N$  matrix models in physics, we look for physical models related to noncommutative probability theory of type B. These turn out to be fermionic matrix-vector models at the double large- $N$  limit. They describe different quantum orbifold spacetimes with boundaries. Their critical exponents coincide with that of ordinary quantum spacetime, but their renormalised tree-level one-boundary amplitudes differ.

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# 1 Introduction

The interplay between noncommutative probability and large- $N$  matrix models is a wonderful example of how mathematics and physics mutually benefit from each other. Noncommutative probability theory (of type A) was originally developed by Voiculescu as a tool to analyse some von Neumann algebras [1]; this analysis was rendered possible by random matrices. It can be shown that a large class of random matrices is asymptotically free as the order of the matrices is taken to infinity. Hence it constitutes a major example of noncommutative probability.

Nonetheless, random matrices show up in physics as well. An important example is Yang–Mills theory. The “infinite-dimensional” Yang–Mills matrix field was dubbed the master field [2]. The master field enjoys greatly simplifying features and is believed to hold the key to a deeper understanding of Yang–Mills theory. Among other examples are quantum gravity and string theory. The large- $N$  expansion of matrix models was found out to be a genus expansion [3]; the dual of the Feynman diagrams of the leading term may be treated as discretised random spherical surfaces of quantum spacetime or string worldsheet [4, 5, 6]. The commonality of random matrices in mathematics and physics triggers an interest in the relationship between noncommutative probability theory of type A and large- $N$  matrix models [7, 8, 9, 10, 11, 12]. It is now possible to think of notions of noncommutative probability in physical terms and describe the master field in algebraic terms.

However, mathematicians have recently realised that other noncommutative probability spaces exist. This is achieved by a reformulation of noncommutative probability theory of type A in terms of non-crossing partitions of type A [13], which, in turn, is related to the Möbius inversion theory in the lattices of non-crossing partitions [14]. Then the fact that other types of non-crossing partitions exist [15] prompted an attempt to develop a “noncommutative probability theory of type B” from non-crossing partitions of type B [16]. Given the intimate relation between noncommutative probability theory of type A and large- $N$  matrix models, naturally we would like to ask if there exist large- $N$  matrix models “of type B”.

Yes, such matrix models exist; they are fermionic matrix-vector models in the double large- $N$  limit. Fermionic matrix (vector) models [17] are models in which the matrix entries (vector components) are Grassman numbers. Their convergent behavior is, in general, better than their bosonic counterparts. Physically, they describe two-dimensional quantum spacetime, random polymers [18, 19, 20], or induced gauge theory [21]. The basic ingredients of the models in this article are  $N_v$ -dimensional vectors of  $N_m \times N_m$  matrices of Grassmann numbers; these are thus simultaneously matrices and vectors. We will take  $N_v$  to infinity first and  $N_m$  to infinity afterwards. Then the dual of the Feynman diagrams of these models describes quantum orbifold spacetime or strings in an orbifold.

We will deal with the physical aspect of fermionic matrix-vector models in this article in detail; the more mathematical aspect will be discussed in future works.

Here is a synopsis of this article. We will review the relationship between noncommutative probability theory of type A and large- $N$  matrix models in Section 2. Then we will introduce non-crossing partitions of type B and a class of fermionic matrix-vector models which generates these partitions in Section 3. In Section 4, we will identify these partitions as random surfaces of a bounded region of the orbifold  $\mathbf{R}^2/\mathbf{Z}_2 \times \mathbf{Z}_2$  and compute the full Green functions (moments) at the double large- $N$  limit. We will also show by means of the Schwinger-Dyson equation that the renormalised tree-level one-boundary amplitude and critical exponent associated with the string susceptibility at the continuum limit are identical to those of ordinary quantum spacetime. In Section 5, we will construct a different class of fermionic matrix-vector models which generates random surfaces of a bounded region of the quantum orbifold  $\mathbf{R}^2/\mathbf{Z}_2$ . We will show that its critical exponent is also the same as that of ordinary quantum spacetime, but it has a different one-boundary amplitude. Finally, we will indicate possible extensions of this work in Section 6.

## 2 Noncommutative Probability & Large- $N$ Matrix Models

A *noncommutative probability space* (of type A) was originally defined by D. Voiculescu [1] as an ordered pair  $(\mathcal{A}, \varphi)$ , where  $\mathcal{A}$  is a complex unital  $C^*$ -algebra and  $\varphi : \mathcal{A} \rightarrow \mathbf{C}$  a linear functional satisfying the normalisation condition that  $\varphi(1) = 1$ . Each element  $a$  of  $\mathcal{A}$  may be considered as a random variable and  $\varphi(a)$  its expectation value.

A useful notion of noncommutative probability theory is the non-crossing cumulant [13]. To define what it is, let us digress for a moment and introduce a few auxiliary notions first. Consider a finite set  $F$  of integers. Let  $p$  be a partition of  $F$ . We write  $n_1 \simeq_p n_2$  if  $n_1$  and  $n_2$  are numbers of  $F$  such that they are in the same block (or cell) of the partition  $p$ . Furthermore,  $p$  is a *non-crossing partition of type A* if  $n_1 \simeq_p n_3$  and  $n_2 \simeq_p n_4$  imply that  $n_2 \simeq_p n_3$  for any four integers  $n_1, n_2, n_3$ , and  $n_4$  in  $F$  such that  $n_1 < n_2 < n_3 < n_4$ . An example of a non-crossing partition of type A is illustrated in Fig.1. The set of all non-crossing partitions of type A of  $F$  is denoted as  $NC^{(A)}(F)$ . If  $F = \{1, 2, \dots, n\}$ , we will abbreviate  $NC^{(A)}(F)$  as  $NC^{(A)}(n)$  as there will be no danger of confusion.

A *non-crossing cumulant* is a multilinear functional  $\kappa_n : \mathcal{A}^n \rightarrow \mathbf{C}$ , where  $n$  is any positive integer, such that

$$\varphi(a_1 a_2 \cdots a_k) = \sum_{p \in NC^{(A)}(k)} \prod_{F \text{ is a block of } p} \kappa_{\text{card}(F)}((a_1, a_2, \dots, a_k)|F). \quad (1)$$

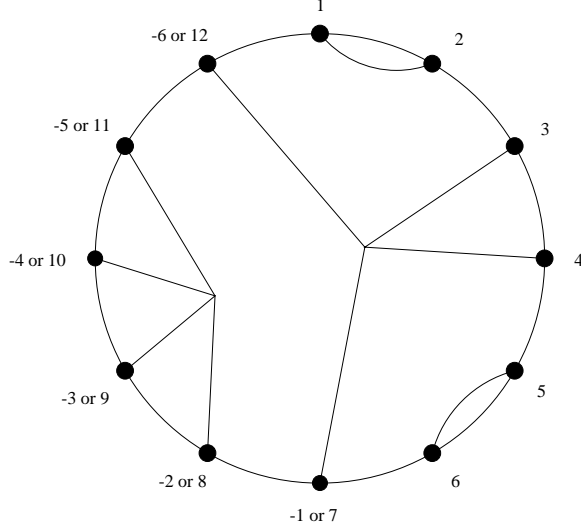


Figure 1: A non-crossing partition of type A but not of type B. The partition of the numbers  $1, 2, \dots, 12$  into the blocks  $\{1, 2\}$ ,  $\{3, 4, 7, 12\}$ ,  $\{5, 6\}$ , and  $\{8, 9, 10, 11\}$  is the non-crossing partition of type A, but the (same) partition of the numbers  $1, 2, \dots, 6, -1, -2, \dots, -6$  into the blocks  $\{1, 2\}$ ,  $\{3, 4, -1, -6\}$ ,  $\{5, 6\}$ , and  $\{-2, -3, -4, -5\}$  is not a non-crossing partition of type B. This may also be regarded as a planar Feynman diagram with each intersection point and the lines connected to it inside the great circle representing a connected Green function.

In this formula, if  $F = \{j_1, j_2, \dots, j_m\}$  such that  $j_1 < j_2 < \dots < j_m$ , then  $\text{card}F$ , the cardinality of  $F$ , is  $m$ , and

$$(a_1, a_2, \dots, a_k) | F := (a_{j_1}, a_{j_2}, \dots, a_{j_m}) \in \mathcal{A}^m.$$

We may apply Eq.(1) recursively to obtain the expressions of all non-crossing cumulants in terms of  $\varphi$ . For example,

$$\begin{aligned} \kappa_1(a_1) &= \varphi(a_1), \\ \kappa_2(a_1, a_2) &= \varphi(a_1, a_2) - \varphi(a_1)\varphi(a_2), \\ \kappa_3(a_1, a_2, a_3) &= \varphi(a_1, a_2, a_3) - \varphi(a_1)\varphi(a_2, a_3) - \varphi(a_2)\varphi(a_1, a_3) \\ &\quad - \varphi(a_3)\varphi(a_1, a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3), \end{aligned} \tag{2}$$

and so on. The significance of non-crossing cumulants lies in the fact that they provide a particularly simple criterion to determine whether a family of unital subalgebras of  $\mathcal{A}$  is free and that the generating series of the non-crossing cumulants, called the R-transform, display simplifying properties [1].

Random matrix models in the large- $N$  limit furnish a nice physical example of noncommutative probability theory of type A [8]. In the simplest case, the random variables are, intuitively speaking, generated by a Hermitian random matrix  $M$  of infinite order, which is alternatively known as the master field. The partition function  $Z_b$  of this random matrix model takes the form

$$Z_b = \int dM \exp(-N \text{Tr} V(M)),$$

where  $N$  is the order of the matrix  $M$  and will eventually be taken to infinity,  $V(M)$  is a polynomial of  $M$ , and  $dM$  is the corresponding measure. The expectation value of the random variable  $M^n$  is

$$\tilde{\phi}(n/2) := \lim_{N \rightarrow \infty} \frac{1}{Z_b} \int dM \frac{1}{N} \text{Tr} M^n \exp(-N \text{Tr} V(M)). \quad (3)$$

$\tilde{\phi}(n/2)$  is called a *full Green function* in physics literature. The non-crossing cumulant  $\kappa_n(M, M, \dots, M)$  is nothing but the connected Green function

$$\lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{Tr} M^n \rangle_c,$$

and a non-crossing partition of type A may be represented by a planar Feynman diagram (Fig.1). Formulae 2 are the expressions of connected Green functions in terms of full Green functions familiar to physicists.

### 3 A Fermionic Matrix-Vector Model

There are other types of non-crossing partitions which are of interest in mathematics [15]. For instance, consider the totally ordered set

$$[\pm n] := \{1 < 2 < \dots < n < -1 < -2 < \dots < -n\}.$$

It is obviously isomorphic to the set

$$[2n] := \{1 < 2 < \dots < 2n\}.$$

Moreover, define the *inversion map* to be a map which maps the integer  $a$  to  $-a$ . A *non-crossing partition of type B* of  $[\pm n]$  is a non-crossing partition of type A of  $[2n]$  such that the partition is invariant under the inversion map. Not all non-crossing partitions of type A are of type B as well; this can be seen in Fig.1. The set of all non-crossing partitions of type B of  $[\pm n]$  is denoted as  $NC^{(B)}(n)$ . A cell of a non-crossing partition of type B is called a *zero-block* if it is invariant under the inversion map; otherwise, it is called a *non-zero-block*. It can be straightforwardly shown that there is at most one zero-block in every non-crossing partition of type B. A partition with a zero-block is illustrated in Fig.2.

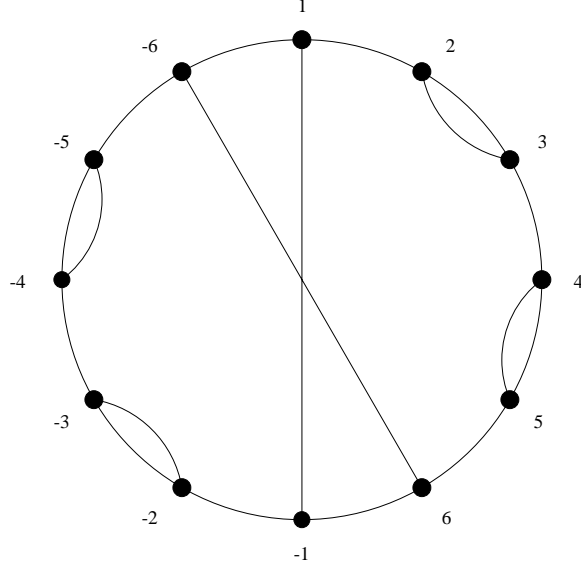


Figure 2: A non-crossing partition of type B. The partition of the numbers  $1, 2, \dots, 6, -1, -2, \dots, -6$  into the blocks  $\{1, 6, -1, -6\}$ ,  $\{2, 3\}$ ,  $\{4, 5\}$ ,  $\{-2, -3\}$ , and  $\{-4, -5\}$  is a non-crossing partition of type B.  $\{1, 6, -1, -6\}$  is a zero-block, and all others are non-zero-blocks.

Associated with non-crossing partitions of type B is a nascent development of a noncommutative probability theory of type B [16] in which the non-crossing cumulants are defined in terms of non-crossing partitions of type B. Given the intimate relationship between noncommutative probability theory and matrix models, it is worthwhile to search for matrix models which are physical realisation of noncommutative probability theory of type B. It turns out that there is such a class of matrix models — the fermionic matrix-vector models in the double large- $N$  limit.

A fermionic matrix-vector model is built out of an  $N_v$ -dimensional vector of matrices  $\Psi_1, \Psi_2, \dots$ , and  $\Psi_{N_v}$ , and another  $N_v$ -dimensional vector of matrices  $\bar{\Psi}_1, \bar{\Psi}_2, \dots$ , and  $\bar{\Psi}_{N_v}$ . Each of these matrices is of order  $N_m$ . Every matrix entry is a Grassmann number. The  $i, j$ -th matrix entry of the matrices  $\Psi_\mu$  and  $\bar{\Psi}_\mu$  are denoted by  $\psi_{\mu ij}$  and  $\bar{\psi}_{\mu ij}$ , respectively. Any two Grassmann numbers anti-commute with each other.

The action of a family of fermionic matrix-vector models takes the form

$$S := N_m \sqrt{N_v} \sum_{\mu=1}^{\infty} \text{Tr} \bar{\Psi}_\mu \Psi_\mu$$

$$\begin{aligned}
& + N_m \sum_{n=1}^{\infty} \frac{c_n}{2n} \sum_{\mu_1, \mu_2, \dots, \mu_{2n}=1}^{N_v} \text{Tr} \left[ (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \cdots \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n}})^2 \right] \\
& + N_m^2 \sum_{n=2}^{\infty} \frac{g_n}{2n} \sum_{\mu_1, \mu_2, \dots, \mu_{2n}=1}^{N_v} \left[ \text{Tr} (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \cdots \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n}}) \right]^2, \quad (4)
\end{aligned}$$

where  $c_1, c_2, c_3, \dots$ , and so on and  $g_2, g_3, g_4, \dots$ , and so on are constant complex numbers. The partition function is given by

$$Z_{N_m, N_v} := \int d\Psi_1 d\bar{\Psi}_1 d\Psi_2 d\bar{\Psi}_2 \cdots d\Psi_{N_v} d\bar{\Psi}_{N_v} \exp S, \quad (5)$$

where the Grassmann integrals are defined as usual in the sense of Brezin [22]. The full Green functions which are of interest to us take the form

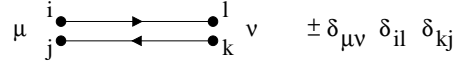
$$G_{N_m, N_v}(n) := \frac{1}{N_m} \sum_{\mu_1, \mu_2, \dots, \mu_{2n}=1}^{N_v} \left\langle \text{Tr} \left[ (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \cdots \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n}})^2 \right] \right\rangle_S,$$

where the subscript  $S$  means that this expectation value is evaluated with respect to the partition function defined by the action  $S$ .

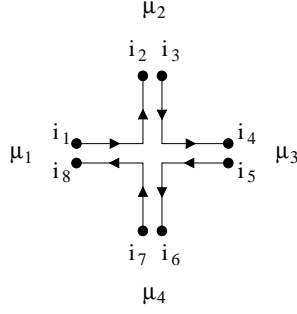
The Feynman rules are analogous to those of  $U(N)$  gauge theory [3]. Some examples are depicted in Fig.3. Note that a *double-end*, i.e, a pair of ends of a double-line representing a propagator or a double-leg of a vertex, represents a row index of an  $N_m \times N_m$  matrix, a column index of the same matrix and an index of an  $N_v$ -dimensional vector. If the subscript of a vector index is odd, then the corresponding double-end originates from  $\bar{\Psi}$ ; otherwise, it originates from  $\Psi$ . For instance, the indices  $\mu_1$  and  $\mu_3$  in the  $c_1$ -vertex originates from  $\bar{\Psi}$ , whereas the indices  $\mu_2$  and  $\mu_4$  originates from  $\Psi$ . As a result, we may connect two double-ends together by a propagator only if the subscript of the vector index of one of the double-ends is odd and the subscript of the vector index of the other double-end is even. For example, we may use a propagator to connect the double-end of a  $c_1$  vertex whose vector index is  $\mu_1$  to the double-end of a  $g_2$  vertex whose vector index is  $\nu_4$ , but we may not use a propagator to connect the same double-end of the  $c_1$  vertex with the double-end of the  $g_2$  vertex whose vector index is  $\nu_3$ . Each of the lines connects a row index to a column index; the connections have nothing to do with vector indices. A fermionic loop may contribute a factor of  $N_m$  or  $-N_m$ . The sign is determined by first principles; two Feynman diagrams which are topologically equivalent after removal of the vector indices may carry opposite signs.

The *double large- $N$  limit* is characterised by the following formulae for the partition function and the Green functions:

$$\begin{aligned}
Z &:= \lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} Z_{N_m, N_v}, \text{ and} \\
G(n) &:= \lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} G_{N_m, N_v}(n). \quad (6)
\end{aligned}$$

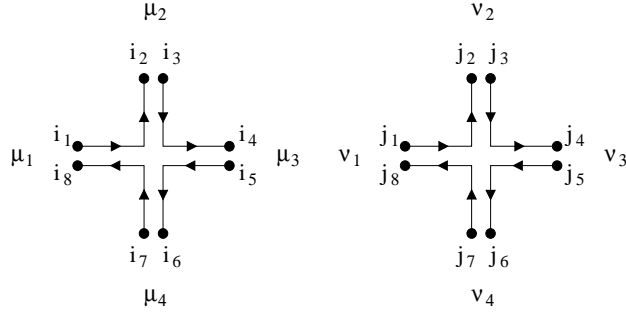


(a) propagator



$$\delta_{\mu_1\mu_3} \delta_{\mu_2\mu_4} \delta_{i_1i_2} \delta_{i_3i_4} \delta_{i_5i_6} \delta_{i_7i_8} \frac{c_1}{2}$$

(b)  $c_1$ -vertex



$$\delta_{\mu_1\nu_1} \delta_{\mu_2\nu_2} \delta_{\mu_3\nu_3} \delta_{\mu_4\nu_4} \delta_{i_1i_2} \delta_{i_3i_4} \delta_{i_5i_6} \delta_{i_7i_8} \delta_{j_1j_2} \delta_{j_3j_4} \delta_{j_5j_6} \delta_{j_7j_8} \frac{g_2}{4}$$

(c)  $g_2$ -double-vertex

Figure 3: *Selected Feynman rules for a fermionic matrix-vector model. Depicted here are (a) a propagator, (b) a vertex with a coefficient  $c_1$ , and (c) a double-vertex with a coefficient  $g_2$ . Each pair of solid circles represents a double-end. Each fermionic loop contributes, up to a sign, a factor of  $N_m$ .*



Note that we take  $N_v$  to infinity *before* taking  $N_m$  to infinity; this ensures that only connected Feynman diagrams will contribute to  $G(n)$ . Moreover, only those Feynman diagrams which satisfy the following condition survive the large- $N_v$  limit:

*if, in a Feynman diagram, a propagator connects together two double-ends whose vector indices are  $\mu_a$  and  $\nu_b$ , then in the same diagram there must be another propagator connecting two other double-ends whose vector indices are  $\mu_a$  and  $\nu_b$ , too.*

Furthermore, only planar Feynman diagrams which survive the large- $N_v$  limit contribute to the double large- $N$  limit. An example of such a Feynman diagram is illustrated in Fig.4. This diagram contributes to Fig.2, which, as a Feynman diagram, each intersection point and the lines connected to it inside the great circle represent a connected Green function, any pair of vector indices  $\mu_k$  of the sources are abbreviated as  $k$  and  $-k$  for any value of  $k$ , and the vector indices of the vertices are suppressed.

## 4 Quantum Orbifold Spacetime

It is well known that the dual of the Feynman diagrams of matrix models associated with noncommutative probability theory of type A may be regarded as discrete random surfaces of two-dimensional quantum gravity. (See, e.g, Refs.[23] and [24] and the references therein.) Does the fermionic matrix-vector model play a role in quantum gravity as well?

A clue may be found in Fig.4. Choose the center of the great circle to be the origin of a two-dimensional Cartesian coordinate system. Point the  $y$ -axis to the index  $i_1$ . Then the figure respects parity transformation  $x \rightarrow -x$  and  $y \rightarrow -y$  about the origin. So does its dual. Though a thoroughly rigorous proof is lacking, it is probably true that every Feynman diagram of  $G(n)$  which contributes to the double large- $N$  limit exhibits the same symmetry. Hence we assert that *the dual of the Feynman diagrams of  $G(n)$  in the double large- $N$  limit may be thought of as quadrangulated random surfaces which respect parity transformation about the origin, i.e, these are random surfaces of a bounded region of the orbifold  $\mathbf{R}^2/\mathbf{Z}_2 \times \mathbf{Z}_2$ .* (c.f. Fig.5.)

Let us study the full Green functions and critical behavior of the simplest fermionic matrix-vector model in which only the coefficients  $c = c_1$  and  $g = g_2$  are non-zero. From Eq.(4), the action is

$$\begin{aligned}
S := & N_m \sqrt{N_v} \sum_{\mu=1}^{\infty} \text{Tr} \bar{\Psi}_{\mu} \Psi_{\mu} + \frac{N_m c}{2} \sum_{\mu_1, \mu_2=1}^{N_v} \text{Tr} \left[ (\bar{\Psi}_{\mu_1} \Psi_{\mu_2})^2 \right] \\
& + \frac{N_m^2 g}{4} \sum_{\mu_1, \mu_2, \mu_3, \mu_4=1}^{N_v} \left[ \text{Tr} (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4}) \right]^2. \tag{7}
\end{aligned}$$

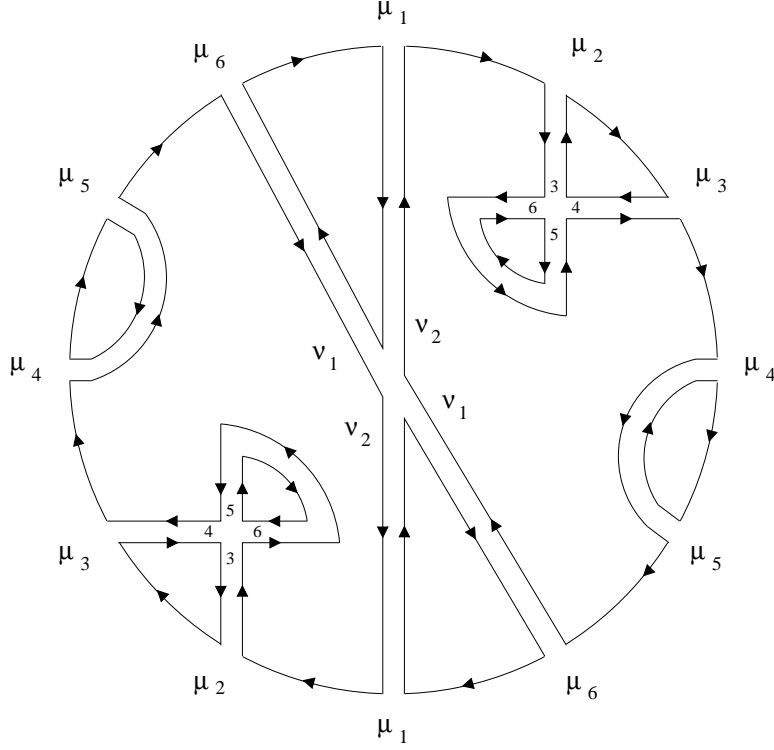


Figure 4: A Feynman diagram of  $G(3)$ . Only vector indices are shown. The numbers 3, 4, 5, and 6 stand for  $\nu_3$ ,  $\nu_4$ ,  $\nu_5$ , and  $\nu_6$ , respectively.

The full Green functions can be obtained by the Schwinger–Dyson approach. Indeed, consider the trivial equation

$$\lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} \frac{1}{N_m^2 \sqrt{N_v}} \sum_{i,j=1}^{N_m} \sum_{\beta=1}^{N_v} \frac{\partial}{\partial \bar{\Psi}_{\beta ij}} \sum_{\alpha_1, \alpha_2, \dots, \alpha_{2n-1}=1}^{N_v} \left\langle \left( \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \Psi_{\beta} \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \right)_{ij} \right\rangle_S = 0 \quad (8)$$

where  $S$  was given in Eq.(7) and  $n$  is any positive integer. Since

$$\lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} \frac{1}{N_m^2 \sqrt{N_v}} \sum_{i,j=1}^{N_m} \sum_{\beta=1}^{N_v} \sum_{\alpha_1, \alpha_2, \dots, \alpha_{2n-1}}^{N_v} \left\langle \left( \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \Psi_{\alpha_{2k}} \cdot \left( \frac{\partial}{\partial \bar{\Psi}_{\beta ij}} \bar{\Psi}_{\alpha_{2k+1}} \right) \Psi_{\alpha_{2k+2}} \cdots \bar{\Psi}_{\alpha_{2n-1}} \Psi_{\beta} \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \right)_{ij} \right\rangle_S$$

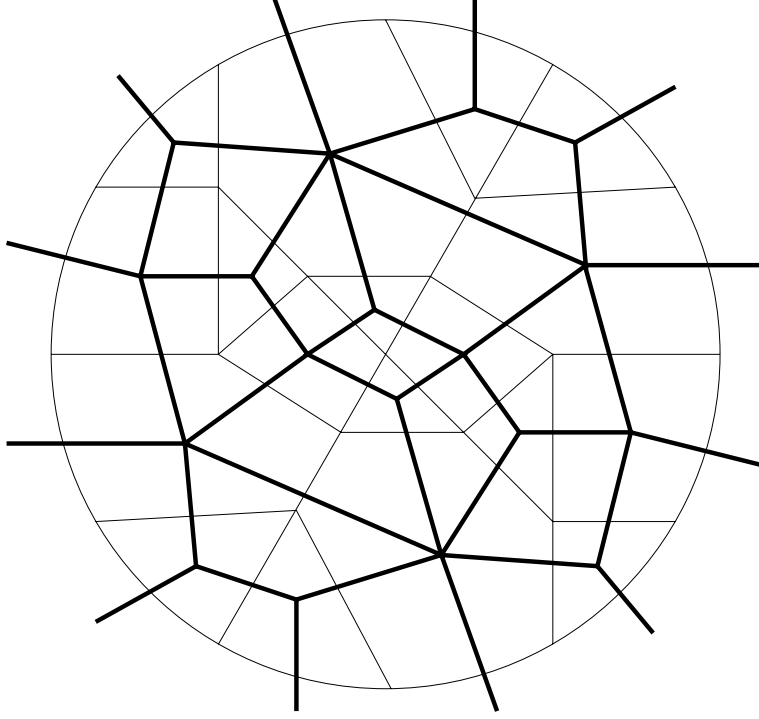


Figure 5: *A discretized random orbifold surface with a boundary. The quadrilaterals paving the surface are in thick lines, and the thin circle forms the boundary. The dual of this surface, drawn in thin lines, is a Feynman diagram of the fermionic matrix-vector model defined in Eq.(7) in the double large- $N$  limit. Note that the “spokes” protruding from the great circle converge to the same point on a spherical surface.*

$$= \tilde{\phi}(k)G(n-k-1),$$

where  $\tilde{\phi}(n)$  was defined in Eq.(3) with  $V(M) = M^2 - (g/4)M^4$  and  $0 \leq k \leq n-2$ ,

$$\begin{aligned} & \lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} \frac{1}{N_m^2 \sqrt{N_v}} \sum_{i,j=1}^{N_m} \sum_{\beta=1}^{N_v} \sum_{\alpha_1, \alpha_2, \dots, \alpha_{2n-1}}^{N_v} \left\langle \left( \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \Psi_{\alpha_{2n-2}} \right. \right. \\ & \quad \left. \cdot \left( \frac{\partial}{\partial \bar{\Psi}_{\beta ij}} \bar{\Psi}_{\alpha_{2n-1}} \right) \Psi_{\beta} \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \right)_{ij} \right\rangle_S = -\tilde{\phi}(n-1), \\ & \lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} \frac{1}{N_m^2 \sqrt{N_v}} \sum_{i,j=1}^{N_m} \sum_{\beta=1}^{N_v} \sum_{\alpha_1, \alpha_2, \dots, \alpha_{2n-1}}^{N_v} \left\langle \left( \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \Psi_{\beta} \right. \right. \end{aligned}$$

$$\cdot \left( \frac{\partial}{\partial \bar{\Psi}_{\beta ij}} \bar{\Psi}_{\alpha_1} \right) \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \Bigg\rangle_{ij} = \tilde{\phi}(n-1),$$

and

$$\begin{aligned} & \lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} \frac{1}{N_m^2 \sqrt{N_v}} \sum_{i,j=1}^{N_m} \sum_{\beta=1}^{N_v} \\ & \sum_{\alpha_1, \alpha_2, \dots, \alpha_{2n-1}}^{N_v} \left\langle \left( \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \Psi_{\beta} \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \Psi_{\alpha_{2k}} \right. \right. \\ & \cdot \left. \left. \left( \frac{\partial}{\partial \bar{\Psi}_{\beta ij}} \bar{\Psi}_{\alpha_{2k+1}} \right) \bar{\Psi}_{\alpha_{2k+2}} \cdots \bar{\Psi}_{\alpha_{2n-1}} \right) \right\rangle_{ij} = \tilde{\phi}(n-k-1)G(k) \end{aligned}$$

for  $1 \leq k \leq n-1$ , it follows from Eq.(8) that the *Schwinger-Dyson equations* are given by

$$-G(1) + c\tilde{\phi}(1) + gG(2) = 0 \quad (9)$$

and

$$2 \sum_{k=0}^{n-2} \tilde{\phi}(k)G(n-1-k) - G(n) + c\tilde{\phi}(n) + gG(n+1) = 0 \quad (10)$$

for  $n = 2, 3, 4, \dots$ , and so on.

Let

$$\phi(z) := \sum_{n=0}^{\infty} \tilde{\phi}(n)z^{2n}$$

be a generating function of the full Green functions of the bosonic matrix  $\phi^4$ -theory. It is well known [25] that

$$\phi(z) = \frac{1}{2z^2} - \frac{g}{2z^4} - \frac{1}{z^2} \left( \frac{1}{2} - g\gamma^2 - \frac{g}{2z^2} \right) \sqrt{1 - 4\gamma^2 z^2}, \quad (11)$$

where

$$\gamma^2 := \frac{1 - \sqrt{1 - 12g}}{6g}. \quad (12)$$

Let

$$\omega_1(z) := \sum_{n=0}^{\infty} G(n)z^{2n}$$

be a generating function of the full Green functions of the fermionic matrix-vector model. It then follows from Eqs.(9) and (10) that

$$\omega_1(z) = 1 + \frac{gG(1) + c - c\phi(z)}{[g - (1 - 2g\gamma^2)z^2] \sqrt{1 - 4\gamma^2 z^2}} z^2. \quad (13)$$

It is clear from Eq.(11) that  $\phi(z)$  has a branch cut between  $-1/(2\gamma)$  and  $1/(2\gamma)$  but is otherwise holomorphic on the complex  $z$ -plane. The usual practice of matrix model analysis is to demand that the holomorphic structure of the generating function be as simple as possible [26, 27, 28, 9, 17, 20]. Moving along the same spirit, we assert that  $\omega_1(z)$  is holomorphic on the whole complex plane except the same branch cut. In particular, the singularities at

$$z = \pm \sqrt{\frac{g}{1-2g\gamma^2}}$$

are removable. Then the numerator of the second term of the right hand side of Eq.(13) vanishes if  $z^2 = g/(1-2g\gamma^2)$ . This fixes  $G(1)$ :

$$G(1) = \frac{c}{g} \phi \left( \sqrt{\frac{g}{1-2g\gamma^2}} \right) - \frac{c}{g}. \quad (14)$$

Let

$$\omega(z) := \frac{1}{z} \omega_1 \left( \frac{1}{z} \right).$$

Then Eqs.(13) and (14) yield

$$\omega(z) = 1 - \frac{cz^2}{2} + \rho(z), \quad (15)$$

where

$$\rho(z) := \frac{cz [2(\gamma^2 + 2) - 3z^2 + 3gz^4]}{6(gz^2 - 1 + 2g\gamma^2) \sqrt{z^2 - 4\gamma^2}} \quad (16)$$

has a branch cut between  $-2\gamma$  and  $2\gamma$ . It is clear from Eq.(16) that  $G(n) = 0$  for any positive value of  $n$  if  $c = 0$ . This is consistent with the well-known fact in fermionic matrix models that all even moments vanish if the model respects chiral transformation [17]

$$\Psi \rightarrow \bar{\Psi} \text{ and } \bar{\Psi} \rightarrow -\Psi.$$

Criticality occurs if the zeros of  $\rho(z)$  coalesce with an end-point of the branch cut. It then follows from Eqs.(16) and (12) that the critical values of  $g$  and  $z$  are

$$g_* = 1/12 \text{ and } z_* = 2\sqrt{2},$$

respectively. Set

$$g = g_* \exp(-a^2 \Lambda_s) \text{ and } z = z_* \exp(a \Lambda_l),$$

where  $a$  is the *cut-off length* and  $\Lambda_s$  and  $\Lambda_l$  are the *continuum bulk and boundary cosmological constants*, respectively. (c.f. Ref.[27].) At the *continuum limit*,  $a$  is small and Eq.(15) becomes

$$\omega(z) \simeq 2ca^{-\frac{1}{2}} w(\Lambda_s, \Lambda_l),$$

where

$$w(\Lambda_s, \Lambda_l) := \frac{1}{(\sqrt{\Lambda_s} + 2\Lambda_l)^{\frac{1}{2}}}$$

may be regarded as the *renormalized tree-level one-boundary amplitude* of the quantum orbifold spacetime. This is different from the corresponding amplitude of ordinary quantum gravity. The scaling behavior of this fermionic matrix-vector model near criticality can be determined as usual by the *string susceptibility*

$$\begin{aligned} \chi &:= \lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} \frac{1}{N_m^4} \frac{\partial^2}{\partial g^2} \langle \log Z_{N_m, N_v} \rangle_S \\ &= \lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} \frac{1}{N_m^2} \frac{\partial}{\partial g} \left\langle \frac{1}{4} \sum_{\mu_1, \mu_2, \mu_3, \mu_4=1}^{N_v} [\text{Tr}(\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4})]^2 \right\rangle_S \\ &= \frac{\partial}{\partial g} (\tilde{\phi}(2)). \end{aligned} \quad (17)$$

The *critical exponent*  $\gamma_{\text{str}}$  is defined by the formula

$$\chi \simeq (g - g_*)^{-\gamma_{\text{str}}}$$

up to a proportionality constant which depends on  $g_*$  only. It then follows from Eqs.(18), (11), and (12) that

$$\gamma_{\text{str}} = -1/2,$$

just like that of ordinary quantum gravity without matter [23].

## 5 Another Quantum Orbifold Spacetime

We may arrange the fermionic matrix-vectors differently to form different models. For instance, consider another family of fermionic matrix-vector models whose actions take the form

$$\begin{aligned} S' &:= N_m \sqrt{N_v} \sum_{\mu=1}^{\infty} \text{Tr} \bar{\Psi}_{\mu} \Psi_{\mu} \\ &+ N_m \sum_{n=1}^{\infty} c'_n \sum_{\mu_1, \mu_2, \dots, \mu_{2n}=1}^{N_v} \text{Tr} (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \cdots \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n}} \\ &\quad \cdot \Psi_{\mu_{2n}} \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n-2}} \bar{\Psi}_{\mu_{2n-3}} \cdots \Psi_{\mu_2} \bar{\Psi}_{\mu_1}) \\ &+ N_m^2 \sum_{n=2}^{\infty} \frac{g'_n}{2n} \sum_{\mu_1, \mu_2, \dots, \mu_{2n}=1}^{N_v} \text{Tr} (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \cdots \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n}}) \\ &\quad \cdot \text{Tr} (\Psi_{\mu_{2n}} \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n-2}} \bar{\Psi}_{\mu_{2n-3}} \cdots \Psi_{\mu_2} \bar{\Psi}_{\mu_1}). \end{aligned} \quad (19)$$

The partition function  $Z'_{N_m, N_v}$  is given by Eq.(5) with  $S$  replaced with  $S'$ . We are interested in the full Green functions which read

$$G'_{N_m, N_v}(n) := \frac{1}{N_m} \sum_{\mu_1, \mu_2, \dots, \mu_{2n}=1}^{N_v} \langle \text{Tr} (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \cdots \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n}} \cdot \Psi_{\mu_{2n}} \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n-2}} \bar{\Psi}_{\mu_{2n-3}} \cdots \Psi_{\mu_2} \bar{\Psi}_{\mu_1}) \rangle_{S'}.$$

The only changes to the Feynman rules are in the vertices. Some vertices are depicted in Fig.6.

The double large- $N$  limit is taken as in Eq.(6) with  $Z$ ,  $Z_{N_m, N_v}$ ,  $G(n)$ , and  $G_{N_m, N_v}(n)$  replaced with their primed versions. A Feynman diagram which survives the double large- $N$  limit is illustrated in Fig.7. If we choose the center of the great circle to be the origin of a two-dimensional Cartesian coordinate system and point the  $y$ -axis to  $\mu_1$ , then the diagram together with its dual respects reflection about the  $y$ -axis. This suggests that *the dual of the Feynman diagrams of  $G'(n)$  in the double large- $N$  limit may be conceived of as discrete random surfaces which respect reflection about the  $y$ -axis, i.e., these are random surfaces of a bounded region of the orbifold  $\mathbf{R}^2/\mathbf{Z}_2$ .* (c.f. Fig.8.)

Let us concentrate on the simplest model in this family and choose only the coefficients  $c = c'_1$  and  $g = g'_2$  to be non-zero. It follows from Eq.(19) that

$$S' := N_m \sqrt{N_v} \sum_{\mu=1}^{\infty} \text{Tr} \bar{\Psi}_{\mu} \Psi_{\mu} + N_m c \sum_{\mu_1, \mu_2=1}^{N_v} \text{Tr} (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \Psi_{\mu_2} \bar{\Psi}_{\mu_1}) + \frac{N_m^2 g}{4} \sum_{\mu_1, \mu_2, \mu_3, \mu_4=1}^{N_v} \text{Tr} (\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4}) \text{Tr} (\Psi_{\mu_4} \bar{\Psi}_{\mu_3} \Psi_{\mu_2} \bar{\Psi}_{\mu_1}). \quad (20)$$

To obtain the full Green functions, consider the trivial equation

$$\lim_{N_m \rightarrow \infty} \lim_{N_v \rightarrow \infty} \frac{1}{N_m^2 \sqrt{N_v}} \sum_{i,j=1}^{N_m} \sum_{\beta=1}^{N_v} \frac{\partial}{\partial \bar{\Psi}_{\beta ij}} \sum_{\alpha_1, \alpha_2, \dots, \alpha_{2n-1}}^{N_v} \langle (\bar{\Psi}_{\alpha_{2n-1}} \Psi_{\alpha_{2n-2}} \cdots \bar{\Psi}_{\alpha_1} \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \Psi_{\beta})_{ij} \rangle_{S'} = 0$$

for any positive integer  $n$ . After some manipulations, this equation leads to the Schwinger–Dyson equation

$$\sum_{k=1}^n \tilde{\phi}(n-k) G'(k-1) + (1+c) G'(n) + g G'(n+1) = 0, \quad (21)$$

where  $n$  is any positive integer and  $\tilde{\phi}(n)$  was defined in Eq.(3) again with  $V(M) = M^2 - (g/4)M^4$ . Introduce the following generating function of the

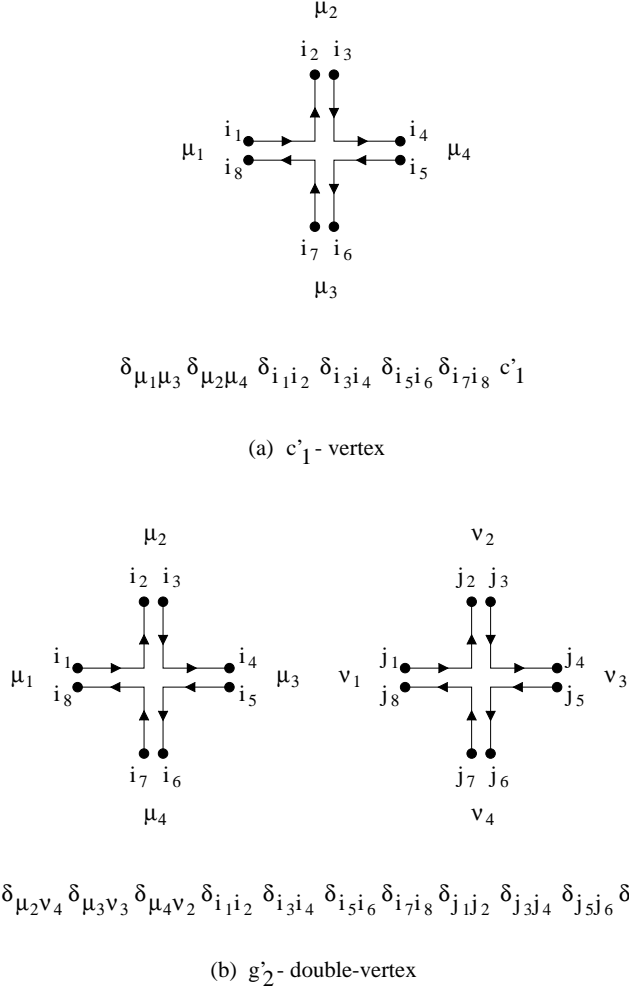


Figure 6: *Selected vertices of the fermionic matrix-vector model defined in Eq.(19). Depicted are (a) a vertex with a coefficient  $c'_1$  and (c) a double-vertex with a coefficient  $g'_2$ .*



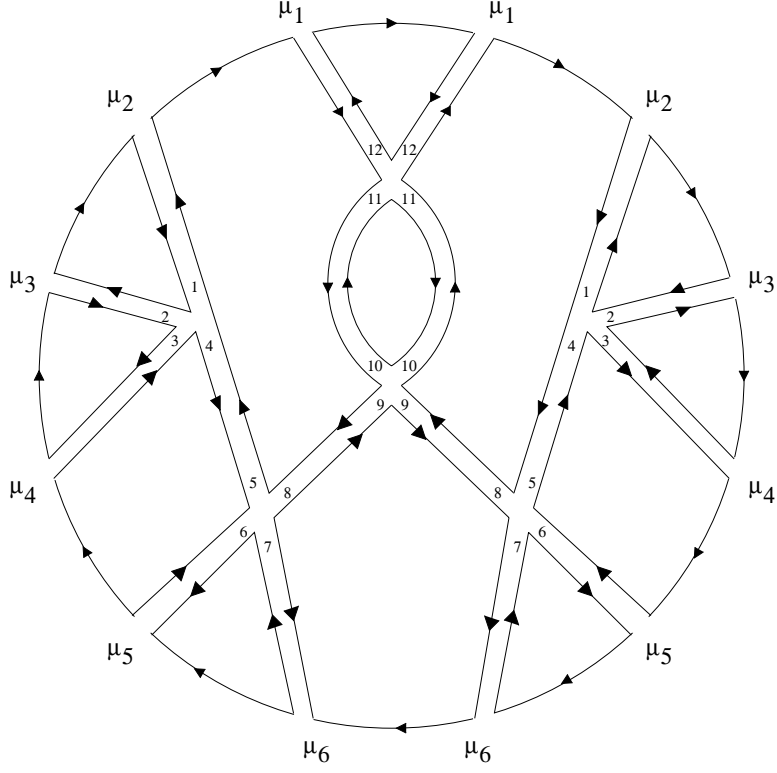


Figure 7: A Feynman diagram of  $G'(3)$ . Only vector indices are shown. The numbers 1, 2, 3, ..., and so on stand for  $\nu_1, \nu_2, \nu_3$ , and so on, respectively.

moments:

$$\omega'_1(z) := \sum_{n=0}^{\infty} G'(n) z^{2n}.$$

Then we may rewrite Eq.(21) as

$$\omega'_1(z) = \frac{g + [1 + c + gG'(1)] z^2}{g + (1 + c)z^2 + z^4\phi(z)},$$

where  $\phi(z)$  was defined in Eqs.(11) and (12). Hence,

$$\omega'_1(z) = \frac{P_1(z^2) \left[ \frac{g}{2} + \left( \frac{3}{2} + c \right) z^2 - \tilde{\rho}(z^2) \right]}{P_3(z^2)}, \quad (22)$$

where

$$P_1(\zeta) := g + [1 + c + gG'(1)] \zeta,$$

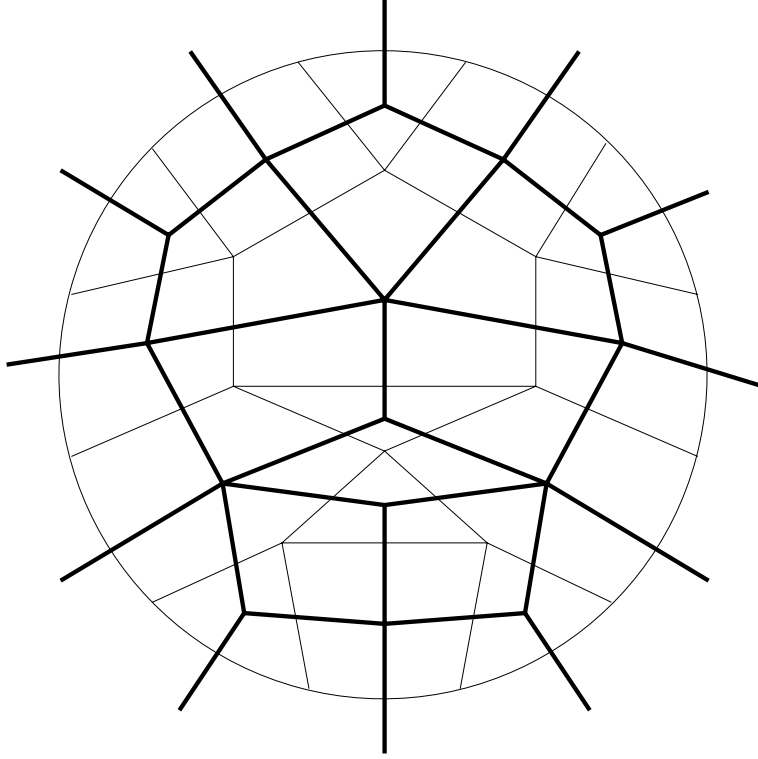


Figure 8: A discretized random orbifold surface with a boundary generated by the fermionic matrix-vector model defined in Eq.(20). This surface respects reflection symmetry instead of parity. the “spokes” protruding from the great circle converge to the same point on a spherical surface.

$$\tilde{\rho}(\zeta) := \left[ \frac{g}{2} + \left( g\gamma^2 - \frac{1}{2} \right) \zeta \right] \sqrt{1 - 4\gamma^2 \zeta},$$

and

$$P_3(\zeta) := \zeta \left[ (2+c)g + (2+3c+c^2-g)\zeta + \left( -\frac{4}{3}g\gamma^2 + \frac{1}{9}\gamma^2 + \frac{8}{9} \right) \zeta^2 \right].$$

Zero and one of the non-zero roots of the cubic polynomial  $P_3(\zeta)$  are roots of  $\tilde{\rho}(\zeta)$ , too. Hence  $G'(1)$  is fixed by the requirement that  $P_1(\zeta)$  vanish if  $\zeta$  is equal to the other non-zero root of  $P_3(\zeta)$ . Then we may read off the full Green functions from  $\omega'_1(z)$ .

At the critical point, the zeros of

$$\rho'(z) := -\frac{P_1(z^2)\tilde{\rho}(z^2)z}{P_3(z^2)}$$

coalesce with an end-point of its branch cut. Thus the critical values of  $g$ ,  $c$ , and  $z$  are

$$g_* = \frac{1}{12}, \quad c_* = -\frac{11}{6}, \quad \text{and} \quad z_* = \frac{1}{2\sqrt{2}}.$$

Near the critical point,

$$g = g_* \exp(-a^2 \Lambda_s), \quad c = c_* \exp(-a^2 \Lambda_c), \quad \text{and} \quad z = z_* \exp(-a \Lambda_l),$$

where  $a$  is small at the continuum limit. Then

$$\rho'(z) \simeq -\frac{1}{4\sqrt{2}} w'(\Lambda_s, \Lambda_l) \sqrt{a},$$

where

$$w'(\Lambda_s, \Lambda_l) := \frac{1}{\Lambda_l} \left( \sqrt{\Lambda_s} - 4\Lambda_l \right) \left( \sqrt{\Lambda_s} + 2\Lambda_l \right)^{\frac{1}{2}}$$

is the renormalised tree-level one-boundary amplitude of this quantum orbifold spacetime. The string susceptibility is defined as in Eq.(17) with  $S$  replaced with  $S'$ . It turns out that

$$\chi = \frac{\partial}{\partial g} \left( \tilde{\phi}(2) \right) \simeq (g - g_*)^{\frac{1}{2}}.$$

Hence we conclude again from Eqs.(11) and (12) that

$$\gamma_{\text{str}} = -1/2.$$

The critical exponent is identical to but the tree-level one-boundary amplitude is different from that of ordinary quantum spacetime.

## 6 Conclusion and Outlook

It is clear from the above calculation that fermionic matrix-vector models may be used to analyse the behavior of different quantum orbifold spacetimes. They have the same critical exponent as that of ordinary quantum spacetime, but the one-boundary amplitudes are already different at the tree level. It is clearly of interest to study the multi-loop amplitudes and double-scaling or even triple-scaling limit of these models as well as their multicritical generalisations and explore the subsequent ramifications in quantum gravity and string theory.

On the more mathematical side, clarification of the relationship between the set of fermionic matrix-vectors of infinite order with the ordered pair of a  $C^*$

algebra and a vector space, constituents of a noncommutative probability theory of type B as defined in Ref.[16], will be desirable. Fermionic matrix-vector models may even be useful in the applications of noncommutative probability of type B to algebraic analysis. (c.f. the application of ordinary random matrix models, the embodiment of noncommutative probability of type A, to the analysis of von Neumann algebras [29].) Furthermore, just like it is interesting to construct rigorous topological or combinatorial arguments to relate the Feynman diagrams of ordinary matrix models with quadrangulated random surfaces [30, 31], it would be desirable to develop a rigorous proof for the identification of the dual of Feynman diagrams of fermionic matrix-vector models with quadrangulated orbifold surfaces and explore the consequences.

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## References

- [1] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free Random Variables*, CRM Monograph Series Vol. 1 (American Mathematical Society, 1992).
- [2] E. Witten, in *Recent developments in gauge theories*, ed. G. 't Hooft et al. (Plenum Press, 1980).
- [3] G. 't Hooft, *A planar diagram theory for strong interactions*, Nucl. Phys. **B 72** (1974) 461—473.
- [4] F. David, *Planar diagrams, two-dimensional lattice gravity and surface models*, Nucl. Phys. **B 257** (1985) 45—58.
- [5] V. A. Kazakov, A. A. Migdal, and I. K. Kostov, *Critical properties of randomly triangulated planar random surfaces*, Phys. Lett. **B 157** (1985) 295—300.
- [6] J. Ambjørn, B. Durhuus, and J. Fröhlich, *Diseases of triangulated random surface models, and possible cures*, Nucl. Phys. **B 257** (1985) 433—449.
- [7] M. R. Douglas, *Stochastic master fields*, Phys. Lett. **B 344** (1995) 117—126 [[hep-th/9411025](#)].
- [8] R. Gopakumar and D. J. Gross, *Mastering the master field*, Nucl. Phys. **B 451** (1995) 379—415 [[hep-th/9411021](#)].
- [9] M. R. Douglas and M. Li, *Free variables and the two matrix model*, Phys. Lett. **B 348** (1995) 360—364 [[hep-th/9412203](#)].

- [10] L. Accardi, I. Ya. Aref'eva, S. V. Kozyrev, and I. V. Volovich, *The master field for large  $N$  matrix models and quantum groups*, Mod. Phys. Lett. **A 10** (1995) 2323—2334 [[hep-th/9503041](#)].
- [11] M. Engelhardt, *Non-relativistic particle in free random gauge background*, Nucl. Phys. **B 481** (1996) 479—510 [[hep-th/9604012](#)].
- [12] A. Agarwal, L. Akant, G. S. Krishnaswami, and S. G. Rajeev, *Collective potential for large  $N$  Hamiltonian matrix models and free Fisher information*, e-print [hep-th/0207200](#).
- [13] R. Speicher, *Multiplicative functions on the lattice of noncrossing partitions and free convolution*, Math. Ann. **298** (1994) 611—628.
- [14] G. Kreweras, *Sur les partitions non-croisées d'un cycle*, Discrete Math. **1** (1972) 333—350.
- [15] V. Reiner, *Non-crossing partitions for classical reflection groups*, Discrete Math. **177** (1997) 195—222.
- [16] P. Biane, F. Goodman, and A. Nica, *Non-Crossing Cumulants of Type B*, e-print [math.OA/0206167](#).
- [17] Yu. Makeenko and K. Zarembo, *Adjoint fermion matrix models*, Nucl. Phys. **B 422** (1994) 237—257 [[hep-th/9309012](#)].
- [18] J. Ambjørn, C. F. Kristjansen, and Yu. Makeenko, *Generalized Penner models to all genera*, Physical Review **D 50** (1994) 5193—5203 [[hep-th/9403024](#)].
- [19] N. Marshall, G. W. Semenoff, and R. J. Szabo, *Critical Behaviour of a fermionic random matrix models at large- $N$* , Phys. Lett. **B 351** (1995) 153—161 [[hep-th/9410214](#)].
- [20] G. W. Semenoff and R. J. Szabo, *Fermionic matrix models*, Int. J. Mod. Phys. **A 12** (1997) 2135—2292 [[hep-th/9605140](#)].
- [21] V. A. Kazakov and A. A. Migdal, *Induced gauge theory at large  $N$* , Nucl. Phys. **B 397** (1993) 214—238 [[hep-th/9206015](#)].
- [22] F. A. Brezin, *The Method of Second Quantization*, Pure and Applied Physics Vol. 24 (Academic Press, 1966).
- [23] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, *2-D gravity and random matrices*, Phys. Rept. **254** (1995) 1—133 [[hep-th/9306153](#)].
- [24] J. Ambjørn, B. Durhuus, and T. Jónsson, *Quantum Geometry*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1997).

- [25] E. Brézin, C. Itzykson, G. Parisi, and J.-B. Zuber, *Planar diagrams*, Comm. Math. Phys. **59** (1978) 35—51.
- [26] S. R. Wadia, *Dyson–Schwinger Equations Approach to the Large- $N$  Limit: Model Systems and String Representation of Yang–Mills Theory*, Phys. Rev. **D 24** (1981) 970—978.
- [27] F. David, *Loop equations and nonperturbative effects in two-dimensional quantum gravity*, Mod. Phys. Lett. **A 5** (1990) 1019—1029.
- [28] M. Staudacher, *Combinatorial solution of the two-matrix model*, Phys. Lett. **B 305** (1993) 332—338 [[hep-th/9301038](#)].
- [29] D. V. Voiculescu, *Free entropy*, Bull. London Math. Soc. **34** (2002) 257—278 [[math.OA/0103168](#)].
- [30] D. Bessis, C. Itzykson, and J. B. Zuber, *Quantum field theory techniques in graphical enumeration*, Adv. in Appl. Math. **1** (1980) 109—157.
- [31] D. M. Jackson, *On an integral representation for the genus series for 2-cell embeddings*, Trans. Amer. Math. Soc. **344** (1994) 755—772.